Int. J. Heat Mass Transfer. Vol. 14, pp. 2143–2146. Pergamon Press 1971. Printed in Great Britain

SHORTER COMMUNICATIONS

MINIMUM THICKNESS OF A DRAINING LIQUID FILM

S. GEORGE BANKOFF*
Argonne National Laboratory, Argonne, Illinois, U.S.A.

(*Received* 20 *August* 1970 *and in revised form* 14 *December* 1970)

NOMENCLATURE

E, sum of surface and kinetic energies;
g, dimensionless function of contact angle, equation (9);
g', acceleration due to gravity;
h, film thickness;
I_1, kinetic energy of rivulet, equation (11);
Q, volumetric discharge rate;
R, radius of curvature of rivulet;
u, streamwise velocity;
$\bar{u}_0$, average velocity of uniform film;
x, streamwise distance;
x_1, rivulet half-width
y, normal distance;
β, g'/ν;
ν, kinematic viscosity;
ψ, function, equation (12);
ρ, density;
σ, surface tension;
Θ, polar coordinate;
θ_0, contact angle.

Subscript
r, rivulet;
o, uniform film.

I. INTRODUCTION

THE FORMATION of dry spots in a draining liquid film is of importance, for example, in the rate of expulsion of sodium from a coolant channel after a fast reactor accident, such as pump failure or power excursion. In this case a vapor slug quickly forms in the heated region, expelling a liquid slug ahead of it. The rate of expulsion as well as re-entry depends, in large measure, on the evaporation from a residual liquid film on the fuel surfaces. Clearly, the conditions under which this film ruptures, with attendant formation of dry spots, are of considerable importance. The burnout phenomenon in boiling reactors is also thought to be closely related to the formation of unwetted wall areas.

In previous work [1] the early stages of instability in a heated draining film were considered via a capillary wave theory, extending the classical Yih–Benjamin [2–4] analysis. It was found that below a critical heat flux, q_{sc}, (which is quite large: 28 cal/cm^2 s for Freon–113; 254 cal/cm^2 s for Na) the heat flow has negligible influence on the initial rate of growth of the most dangerous (fastest-growing) wavelength. Above this critical flux, however, rapid destabilization occurs. This small-perturbation theory, however, can at best give only a rough estimate of the time required to form a thin spot, and gives no information concerning the formation of dry spots. For the latter problem the contact angle is an important variable which does not enter into the capillary wave analysis.

The stability of dry spots in a draining film has been principally examined from two points of view, both due originally to Hartley and Murgatroyd [5]. The former examines the vicinity of the stagnation point at the leading edge of a dry patch in a liquid film draining steadily down a heated wall. By means of momentum balances over a suitable control volume, the stationary condition for the triple interface is developed. Subsequent investigations have taken into account surface shear forces and surface tension variations [6–8]. This approach takes into account the contact angle, predicting increased dry-spot stability as the contact angle is increased, as might be expected. On the other hand, no consideration is given to the question of how a semi-infinite dry patch forms in a uniform liquid film, nor of the details of the liquid flow except in the vicinity of the tip of the dry patch.

Murgatroyd and Hartley [5] further postulated that the sum of the kinetic and surface energy flows of the unbroken film was minimized at the critical thickness for stability. Their approach suffers from the disadvantage of omitting consideration of surface wettability, as evidenced by contact angle, and has not been extensively pursued. More recently, an alternative approach to the problem of thermocapillary-induced breakdown of a draining film was given by Simon and Hsu [9], based upon the postulate of steady lateral

* Present address: Chemical Engineering Department; Northwestern University; Evanston, Illinois.

surface tension and film thickness variations. Their treatment of the mass and thermal energy conservation equation is, however, open to considerable question.

The present work was motivated by a consideration of the short-time stability of thin heated films on vertical walls, as would be expected to occur in the coolant expulsion problem. Apart from the conceptual difficulties outlined above, the former (rewetting of a semi-infinite dry patch) approach has here the disadvantage of giving no information concerning the fraction of the wall covered by liquid. The approach adopted here is therefore to employ the Hartley–Murgatroyd energy criterion in an entirely different context, which allows the crucial factor of surface wettability to be taken into account. Briefly, it assumed that the liquid film will break up into rivulets when the total mechanical (kinetic plus surface) energy per unit volume is the same in the two configurations. For smaller flows, therefore, energy considerations favor a break-up into parallel rivulets.

II. ANALYSIS

Consider, first of all, a uniform liquid film, of thickness, h_0, draining steadily down a vertical wall. It is well-known that the velocity profile (assuming zero surface shear stress) is given by:

$$u_0(y) = \beta\left(\frac{y^2}{2} - yh_0\right); \quad \beta = \frac{g}{\nu} \tag{1}$$

where $u_0(y)$ is the liquid velocity at a distance y from the wall; ν is the kinematic viscosity; and g is the gravitational force constant. The total mechanical energy per unit volume is then proportional to the sum of the kinetic and surface energies

$$E_0(h_0) = \frac{\rho}{2}\int_0^{h_0} u_0^2\,\mathrm{d}y + \sigma = \frac{\rho\beta^2 h_0{}^5}{15} + \sigma \tag{2}$$

upon employing equation (1). Here σ is the surface tension,

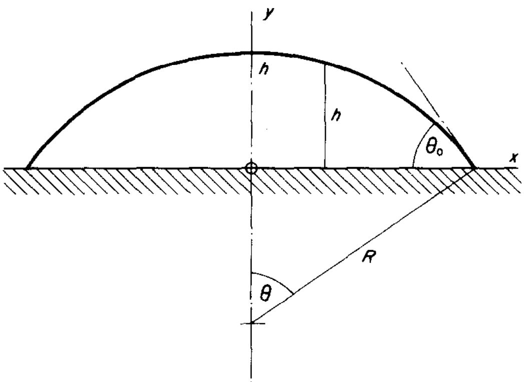

FIG. 1.

considered to be constant, to the present approximation. The total discharge rate per unit width is further given by

$$Q' \equiv \bar{u}_0 h_0 = \frac{\beta h_0{}^3}{3} \tag{3}$$

where $\bar{u}_0$ is the average liquid velocity.

Alternatively, consider the same flow in the form of rivulets whose cross-section is a segment (Fig. 1) of a circle of radius R, given parametrically by $h = h(x)$, where h is the film thickness at a horizontal distance, x, from the center. In terms of the polar subtended angle, θ,

$$h(x) = R(\cos\theta - \cos\theta_0); \quad 0 \leqslant x \leqslant x_1 \tag{4}$$

where θ_0 is the liquid contact angle, and $0 \leqslant \theta \leqslant \theta_0$. We now approximate the velocity at each point by:

$$u(x, y) = \beta\left(\frac{y^2}{2} - yh(x)\right). \tag{5}$$

This assumes that the velocity profile in each slice of width $\mathrm{d}x$ and thickness $h(x)$ is that of a uniform draining film of the same thickness.

The volumetric discharge rate of the rivulet is then

$$Q = 2\int_0^{x_1}\int_0^{h(x)} u(x, y)\,\mathrm{d}y\,\mathrm{d}x = \frac{2\beta R^4}{3}\Phi(\theta_0) \tag{6}$$

upon employing equations (3) and (4), where

$$\Phi(\theta_0) \equiv \int_0^{\theta_0} (\cos\theta - \cos\theta_0)^3 \cos\theta\,\mathrm{d}\theta \tag{7}$$

and x_1, the rivulet half-width, is related to the contact angle and rivulet radius by

$$x_1 = R\sin\theta_0. \tag{8}$$

At the critical film thickness the two discharge rates are equal, whence from equations (3), (6) and (8)

$$\left(\frac{h_0}{R}\right)^3 = \frac{\Phi(\theta_0)}{\sin\theta_0} \equiv g(\theta_0) \qquad \theta_0 \leqslant \frac{\pi}{2}. \tag{9}$$

The sum of the kinetic and surface energies per unit width of the rivulet is

$$E_r = \frac{1}{x_1}(I_1 + \theta_0\sigma R) \tag{10}$$

where the kinetic energy term, from equations (4) and (5), is

$$I_1 \equiv \tfrac{1}{2}\int_0^{x_1}\int_0^{h(x)} \rho u^2\mathrm{d}y\,\mathrm{d}x = \frac{\rho\beta^2 R^6}{15}\psi(\theta_0); \tag{11}$$

$$\psi(\theta_0) \equiv \int_0^{\theta_0} (\cos\theta - \cos\theta_0)^5 \cos\theta\,\mathrm{d}\theta. \tag{12}$$

Setting $E_r = E_0$, we have a second relation which determines the minimum film thickness, h_0:

$$\frac{\rho\beta^2 R^5\psi(\theta_0)/15 + \theta_0\sigma}{\sin\theta_0} = \frac{\rho\beta^2 h_0{}^5}{15} \qquad (13)$$

whence, upon employing equation (9):

$$\frac{\rho\beta^2 h_0{}^5}{15\sigma} = \frac{\theta_0 - \sin\theta_0}{\sin\theta - [g(\theta_0)]^{-5/3}\psi(\theta_0)}. \qquad (14)$$

The integrals can be evaluated directly, giving:

$$\psi(\theta_0) = \theta_0\left(\frac{5}{16} + \frac{15}{4}\cos^2\theta_0 + \frac{5}{2}\cos^4\theta_0\right)$$

$$- \sin\theta_0\left(\frac{113}{48}\cos\theta_0 + \frac{49}{8}\cos^3\theta_0 + \frac{1}{6}\cos^5\theta_0\right) \qquad (15)$$

and

$$g(\theta_0) = -\tfrac{1}{4}\cos^3\theta_0 - \frac{13}{8}\cos\theta_0 + \frac{15}{8}\frac{\theta_0}{\sin\theta}$$

$$- \frac{3}{2}\theta_0\sin\theta_0. \qquad (16)$$

III. RESULTS

Figure 2 shows the minimum stable film thickness, h_o, calculated from equation (14), for sodium, Freon-113 and water, all at atmospheric pressure and saturation temperature, as a function of contact angle. The sodium and Freon lines are remarkably (and fortuitously) close, lending support to the idea of using Freon-113 as a simulant in studying sodium transient two-phase heat transfer in connection with fast reactor safety analyses. For well-wetted walls ($\theta_0 < 15°$), $h_0 \sim 2 \times 10^{-4}$ cm for sodium, which is far smaller than the drainage films of thickness 0·07–0·25 mm observed by Spiller *et al.* [10] in transient voiding of a sodium column. In a typical loss-of-flow reactor accident, the maximum evaporation rate of the sodium film is of the order of 0·1 cm/s of liquid, and the expulsion time is of the order of 0·1 s. One concludes that a well-wetted wall will probably maintain an unbroken film in contact with it during at least the early expulsion period.

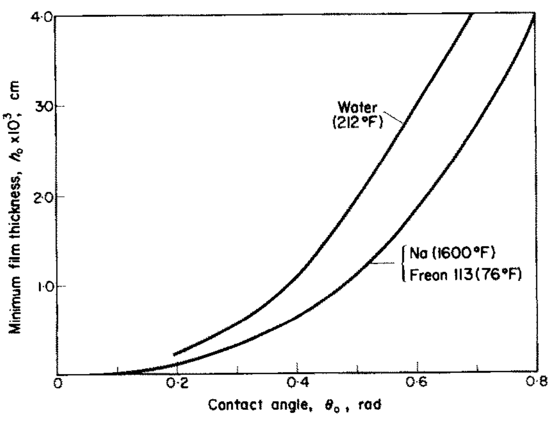

FIG. 2.

It is of some interest to note that the predicted minimum film thickness increases rapidly for contact angles in excess of 45°, in view of the singularity in equation (14). For these thicker films the approximate velocity profile is in greater error (the approximation holding closely when the slope of the rivulet surface is everywhere small), and the more laborious calculation involving the exact velocity profile should be employed, as indicated in Appendix I.

ACKNOWLEDGEMENTS

Mr. J. P. Dalcorso performed the numerical computations and evaluated the trigonometric integrals. This work was conducted under the auspices of the U.S. Atomic Energy Commission.

REFERENCES

1. S. G. Bankoff, Stability of liquid flow down a heated inclined plane, *Int. J. Heat Mass Transfer* **14**, 377–385 (1971).
2. C. S. Yih, Stability of parallel laminar flow with a free surface, *Proc. 2nd U.S. Congress Appl. Mech.*, A.S.M.E., pp. 623–628 (1955).
3. T. B. Benjamin, Wave formation in laminar flow down an inclined plane, *J. Fluid Mech.* **2**, 554–74 (1957).
4. C. S. Yih, Stability of liquid flow down an inclined plane, *Physics Fluids* **6**, 321 (1963).
5. D. E. Hartley and W. Murgatroyd, Criteria for the break-up of thin liquid layers flowing isothermally over a solid surface, *Int. J. Heat Mass Transfer* **7**, 1003 (1964).
6. N. Zuber and F. W. Staub, Stability of dry patches forming in liquid films flowing over heated surfaces, *Int. J. Heat Mass Transfer* **9**, 897–905 (1966).
7. G. D. McPherson, Axial stability of dry patch formed in dryout of two-phase annular flow, *Canadian Congress Appl. Mechanics*, Laval University, Quebec (May 1967).
8. W. Murgatroyd, Role of shear and form forces in stability of dry patch in two-phase film flow, *Int. J. Heat Mass Transfer* **8**, 297 (1965).
9. F. F. Simon and Y. Y. Hsu, Thermocapillary induced breakdown of falling liquid film, NASA TN D-5624, 37 (January 1970).
10. K. H. Spiller, G. Grass and D. Perschke, Überhitzung und Einzelblasenejektion bei der Verdampfung von stagnierenden Flüssigmetallen, *Atomkernenergie* **12**, 111 (1967).

APPENDIX

Velocity Profile in a Draining Rivulet

The equation of motion, assuming steady, parallel flow, is

$$\frac{\mu}{\rho}\nabla^2 v + g = 0; \quad v = v(x, y) \qquad (1A)$$

subject to the boundary conditions:

$$v(x, 0) = 0 \quad -x_1 \leqslant x \leqslant x_1 \tag{2A}$$

$$\frac{\partial v}{\partial n} = 0: \quad (x^2 + y^2)^{\frac{1}{2}} = R: \quad y \geqslant 0. \tag{3A}$$

Letting $z = x + iy$, and introducing coaxal coordinates

$$\zeta = \xi + i\eta = \frac{z + x_1}{z - x_1} \tag{4A}$$

we find that

$$\nabla^2 v = 4\frac{\partial^2 v}{\partial z \partial \bar{z}} = \frac{4x_1^2}{|\sinh^2 \zeta|^2}\frac{\partial^2 v}{\partial \zeta \partial \bar{\zeta}} = \frac{-g}{\nu} \tag{5A}$$

subject to

$$\frac{\partial v}{\partial \xi} = 0: \quad \xi = \pi - \theta_0 \tag{6A}$$

$$v = 0: \quad \xi = \pi.$$

Solution of this problem can be effected numerically, or analytically in terms of a complete set of orthogonal functions.

Int. J. Heat Mass Transfer. Vol. 14, pp. 2146–2152. Pergamon Press 1971. Printed in Great Britain

THE DISPERSION OF MATTER IN TURBULENT SHEAR FLOW

KEMAL M. ATESMEN

Department of Civil Engineering, Colorado State University, Fort Collins, Colorado, U.S.A.

(*Received* 10 *February* 1971 *and in revised form* 29 *March* 1971)

INTRODUCTION

IN THE past fifteen years considerable progress has been recorded by researchers concerning the dispersion of matter in turbulent shear flows. The practical applications of this literature to flow metering in pipes and natural streams, as well as pollution control studies and other engineering problems, are numerous. The formulation of the problem involves seeking solutions for specific initial conditions of the transient diffusion equation. The analyses applied to fully developed channel and pipe flows are based on eddy diffusivity approximations using Reynolds' analogy and semi-empirical solutions for the mean flow field.

The purpose of this paper is to present a general method for finding solutions to the longitudinal dispersion problem. A specific example is presented in detail where both experimental data and an earlier analysis are available for comparison. It is concluded that the analytical methodology proposed here not only has a suitable mathematical formalism, but that it also provides a finite algorithm for numerical solution with advantages over previous methods.

PRESENT ANALYSIS

Aris [1] published a method for solving longitudinal dispersion problems which has been successfully applied by several authors in the last three years. Consider a fully developed channel flow of an incompressible fluid where the time averaged concentration C of the dispersing matter may be described by

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \varepsilon_x\frac{\partial^2 C}{\partial x^2} + \frac{\partial}{\partial y}\left(\varepsilon_y\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varepsilon_z\frac{\partial C}{\partial z}\right) \tag{1}$$

where t is the dispersion time, x, y, z, are the space coordinates in the longitudinal, vertical and lateral directions, respectively, and ε is the eddy diffusivity for mass transfer.

To facilitate comparisons, the notation and dimensionless parameters introduced by Aris [1] will be incorporated into equation (1). Following Aris, define the local velocity as

$$U(y, z) \equiv \bar{U}[1 + \chi(y, z)] \tag{2}$$

where $\chi(y, z)$ is a function which describes the variation of velocity in the cross section, and the local eddy diffusivity as

$$\varepsilon_x = \varepsilon_y = \varepsilon_z \equiv D\psi(y, z) \tag{3}$$

where D is the average value of the eddy diffusivity in the cross section and $\psi(y, z)$ is a function describing the distribution of eddy diffusivity.* Using these definitions, equation (1) becomes

* The present analysis assumes isotropic turbulence structure, but the extension to non-isotropic turbulent flows could be achieved by redefining local eddy diffusivities, namely $\varepsilon_x = D_x\psi_x(y,z)$, $\varepsilon_y = D_y\psi_y(y,z)$ and $\varepsilon_z = D_z\psi_z(y,z)$.